Unique positive solution for an alternative discrete Painlevé I equation

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Abstract

We show that the alternative discrete Painlevé I equation (alt-d$\mathrm{P_I}$) has a unique solution which remains positive for all $n \geq 0$. Furthermore, we identify this positive solution in terms of a special solution of the second Painlevé equation ($\mathrm{P_{II}}$) involving the Airy function $\mathrm{Ai}(t)$. The special-function solutions of $\mathrm{P_{II}}$ involving only the Airy function $\mathrm{Ai}(t)$ therefore have the property that they remain positive for all $n \geq 0$ and all $t \geq 0$, which is a new characterization of these special solutions of $\mathrm{P_{II}}$ and alt-d$\mathrm{P_I}$.

1 Introduction

We will investigate the system of two nonlinear equations

$$a_n + a_{n+1} = b_n^2 - t, \tag{1.1a}$$

$$a_n(b_n + b_{n-1}) = n, \tag{1.1b}$$

which is known as the *alternative discrete Painlevé I equation* (alt-d$\mathrm{P_I}$) [7]. These equations arise, for example, when one wants to find the recurrence coefficients of orthogonal polynomials with an exponential cubic weight; see, e.g. [2, 15]. They also arise when one wants to compute the recurrence coefficients of multiple orthogonal polynomials with an exponential cubic weight [6]. The equations also give relations between solutions of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 2y^3 - 2ty - 2\alpha, \tag{1.2}$$

for different values of the parameter α [4, 11]. Equation (1.2) is equivalent to the second Painlevé equation ($\mathrm{P_{II}}$)

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha, \tag{1.3}$$

through the scaling $w(z) = -2^{-1/3}\, y(t)$, with $z = -2^{1/3}\, t$. We note that if we solve (1.1b) for a_n and substitute this in (1.1a), then we obtain

$$\frac{n+1}{b_n + b_{n+1}} + \frac{n}{b_n + b_{n-1}} = b_n^2 - t. \tag{1.4}$$

The main result in this paper is the following.

Theorem 1.1. *For nonnegative values of t, there exists a unique solution of* (1.1) *with $a_0(t) = 0$ for which $a_{n+1}(t) > 0$ and $b_n(t) > 0$ for all $n \geq 0$, corresponding to the initial value*

$$b_0(t) = -\mathrm{Ai}'(t)/\mathrm{Ai}(t), \tag{1.5}$$

where $\mathrm{Ai}(t)$ *is the Airy function.*

The corresponding result for the nonlinear recurrence (1.4) is

Theorem 1.2. *For nonnegative values of t, there exists a unique solution of* (1.4) *for which $b_n(t) > 0$ for all $n \geq 0$, corresponding to the initial value* (1.5).

We observe that the value of $b_1(t)$ is fixed once $a_0(t)$ and $b_0(t)$ are known. Specifically one has $b_1(t) = -b_0(t) + 1/[b_0^2(t) - t]$. Hence the solutions of (1.4) only depend on *one* free parameter $b_0(t)$ even though it is a recurrence relation of second order. This is reflected in Theorem 1.1 by the initial value $a_0(t) = 0$. It is important to note that the initial value (1.5) involves *only* $\mathrm{Ai}(t)$, rather than a linear combination of the Airy functions $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$.

Uniqueness of positive solutions for non-linear recurrence relations was already studied earlier for the equation

$$x_n(x_{n+1} + x_n + x_{n-1}) = n, \tag{1.6}$$

which is known as a discrete Painlevé I equation [8, 14] and which appears naturally when one wants to find the recurrence coefficients of the orthogonal polynomials with the exponential weight $\exp(-x^4)$ on the real line, the so-called *Freud weight*. Lew and Quarles [13], Nevai [16], and Bonan and Nevai [3] showed that there exists a unique solution of (1.6) with $x_0 = 0$ for which $x_n > 0$ for all $n \geq 1$ and this solution corresponds to $x_1 = 2\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})$. This result was recently also proved for a family of non-linear recurrence relations generalizing (1.6), see [1].

The fact that there is a unique solution of (1.6), or equivalently for (1.4), means that it is not wise to compute this positive solution starting from the initial values $a_0(t) = 0$ and $b_0(t) = -\mathrm{Ai}'(t)/\mathrm{Ai}(t)$, because a small error in b_0 means that one will not be generating the positive solution and after some time there will be negative $b_n(t)$. In §2 we will give a method to compute the positive solution $(b_n)_{n\geq 0}$ by means of a fixed point algorithm using a contraction acting on infinite sequences. That method is stable and is very suitable to compute the positive solution numerically.

One can easily obtain the asymptotic behaviour of the positive solution of (1.6). This was already proved in [6], but we repeat the proof here for completeness.

Corollary 1.3. *For the positive solution of* (1.6) *one has* $\lim_{n\to\infty} a_n/n^{2/3} = \frac{1}{2}$ *and* $\lim_{n\to\infty} b_n/n^{1/3} = 1$.

Proof. The positivity of the a_n and (1.1a) imply that $a_n \leq b_n^2$. The positivity of the b_n and (1.1b) imply $a_n b_n \leq n$. Together this implies $a_n \leq n^2/a_n^2$ so that $a_n \leq n^{2/3}$. From (1.1a) it then follows that $b_n^2 \leq n^{2/3} + (n+1)^{2/3}$. We can then define

$$B_1 = \liminf_{n\to\infty} b_n/n^{1/3}, \qquad B_2 = \limsup_{n\to\infty} b_n/n^{1/3}.$$

If we take a subsequence such that $b_n/n^{1/3} \to B_1$, then from (1.4) one finds

$$B_1^2 \geq 2/(B_1 + B_2).$$

In a similar way, by taking a subsequence such that $b_n/n^{1/3} \to B_2$, it follows from (1.4) that

$$B_2^2 \leq 2/(B_1 + B_2).$$

Together this implies that $B_2^2 \leq B_1^2$, and since one always has $B_1 \leq B_2$, it follows that $B_1 = B_2$ so that $b_n/n^{1/3}$ converges. From (1.1b) it then also follows that $a_n/n^{2/3}$ converges. Taking limits in (1.1a) and (1.1b) then easily gives the values of these limits. □

This paper is organised as follows. In §2 we discuss the uniqueness of b_n and discuss its behaviour in §3. In §4 we discuss the relationship with solutions of $\mathrm{P_{II}}$ (1.3). It is well known that $\mathrm{P_{II}}$ (1.3) has special-function solutions expressible in terms Airy functions [4, 10, 17]. Using these we give explicit expressions for the solutions of (1.1) and (1.4) in terms of the Airy function $\mathrm{Ai}(t)$ in §5. The special-function solutions of $\mathrm{P_{II}}$ involving *only* the Airy function $\mathrm{Ai}(t)$ therefore have the property that they remain positive for all $n \geq 0$ and all $t \geq 0$, which is a new characterization of these special solutions of $\mathrm{P_{II}}$ and alt-d$\mathrm{P_I}$.

2 Proof of the uniqueness

In this section we will prove Theorem 1.2. A positive solution $(b_n)_{n\geq 0}$ implies that $a_0 = 0$ and $a_n > 0$ for $n \geq 1$ by using (1.1b), so that Theorem 1.1 follows as well. The idea of the proof is to construct a mapping T on the space $\mathbb{R}_+^{\mathbb{N}}$ of positive sequences $(x_n)_{n\geq 0}$ and to show that this is a contraction on a complete set in this space. The unique fixed point will be the desired positive solution of (1.4).

Let $(x_n)_{n\geq 0}$ be an infinite sequence with $x_n > 0$ for all $n \geq 0$. We define a new sequence $((\mathrm{T}x)_n)_{n>0}$ implicitly by

$$\frac{n+1}{(\mathrm{T}x)_n + x_{n+1}} + \frac{n}{(\mathrm{T}x)_n + x_{n-1}} = (\mathrm{T}x)_n^2 - t, \qquad n \geq 1 \tag{2.1}$$

and

$$\frac{1}{(\mathrm{T}x)_0 + x_1} = [(\mathrm{T}x)_0]^2 - t. \tag{2.2}$$

Observe that $(\mathrm{T}x)_n$ is a solution of

$$\frac{n+1}{y + x_{n+1}} + \frac{n}{y + x_{n-1}} = y^2 - t. \tag{2.3}$$

If $x_{n+1} > 0$ and $x_{n-1} > 0$, then the left hand side of (2.3) is a positive and decreasing function of $y \in (0, \infty)$ and the right hand side of (2.3) is obviously increasing on $[0, \infty)$ so that (2.3) has a unique positive solution and therefore for $n \geq 0$ we set $(\mathrm{T}x)_n$ to be the positive solution of (2.3). Note that the left hand side requires x_{n+1} and x_{n-1} for $n \geq 1$ but only requires x_1 for $n = 0$. Equation (2.3) corresponds to a quartic equation in y for $n \geq 1$ and a cubic equation if $n = 0$. There is a negative solution between $-x_{n+1}$ and $-x_{n-1}$ when $n \geq 1$ and two complex conjugate solutions or two real negative solutions which will not be used in the remainder of this paper.

We will now give some properties of this mapping T.

Lemma 2.1. *If $x_n > 0$ for all $n \geq 0$, then $0 < (\mathrm{T}x)_n \leq B_n(t)$ with*

$$B_n(t) = (n + \tfrac{1}{2})^{1/3} \left\{ \left[1 - \sqrt{1 - \frac{t^3}{27(n+\frac{1}{2})^2}} \right]^{1/3} + \left[1 + \sqrt{1 - \frac{t^3}{27(n+\frac{1}{2})^2}} \right]^{1/3} \right\}. \tag{2.4}$$

Proof. Since $x_{n+1} > 0$ and $x_{n-1} > 0$, it follows from (2.1)

$$x_n = (\mathrm{T}x)_n = \sqrt{t + \frac{n+1}{(\mathrm{T}x)_n + x_{n+1}} + \frac{n}{(\mathrm{T}x)_n + x_{n-1}}} \leq \sqrt{t + \frac{2n+1}{(\mathrm{T}x)_n}}$$

and therefore

$$(\mathrm{T}x)_n \Big\{ \big[(\mathrm{T}x)_n \big]^2 - t \Big\} \leq 2n + 1.$$

The roots of the equation $y(y^2 - t) = 2n + 1$ can be written as:

$$y_1 = (n + \tfrac{1}{2})^{1/3} \left\{ \left[1 - \sqrt{1 - \frac{t^3}{27(n+\frac{1}{2})^2}} \right]^{1/3} + \left[1 + \sqrt{1 - \frac{t^3}{27(n+\frac{1}{2})^2}} \right]^{1/3} \right\}$$

$$\begin{aligned} y_{2\pm} = -\frac{(n+\frac{1}{2})^{1/3}}{2^{1/3}3} \Bigg\{ & (1 \pm \mathrm{i}\sqrt{3}) \left[1 - \sqrt{1 - \frac{t^3}{27(n+\frac{1}{2})^2}} \right]^{1/3} \\ & + (1 \mp \mathrm{i}\sqrt{3}) \left[1 + \sqrt{1 - \frac{t^3}{27(n+\frac{1}{2})^2}} \right]^{1/3} \Bigg\} \end{aligned}$$

In fact $y_1 = (n+\frac{1}{2})^{1/3} Q(w)$, with $w = \frac{1}{3} t (n+\frac{1}{2})^{-2/3}$, where

$$Q(w) = \left(1 - \sqrt{1-w^3}\right)^{1/3} + \left(1 + \sqrt{1-w^3}\right)^{1/3}. \tag{2.5}$$

Notice that for $w > 1$ it follows that $Q(w) = 2\sqrt{w}\cos\left[\frac{1}{3}\arccos(w^{-3/2})\right]$. Hence, $Q(w)$ is always positive for any real value of w, in fact it is the unique positive solution of the equation

$$Q(w)\left[Q^2(w) - 3w\right] = 2. \tag{2.6}$$

As a consequence, y_1 is positive for any real value of t (and any integer n). The roots $y_{2\pm}$ are either a pair of complex conjugate numbers, when $(2n+1)^2 > \frac{4}{27}t^3$, or two negative numbers, when $(2n+1)^2 < \frac{4}{27}t^3$. The case where eventually $(2n+1)^2 = \frac{4}{27}t^3$ would produce a positive single root y_1 and a double negative root y_{1^-}. Hence, no matter the value of t, there will be a unique positive root, which is $y_1 = B_n(t)$. Now the positivity of $(\mathrm{T}x)_n$ yields the required bounds. $\square$

Let us write $B_n(t) = R\left(n+\frac{1}{2}, t\right)$ with $R(z,t) = z^{1/3} Q\left(t/(3z^{2/3})\right)$, where $Q(w)$ is the function given by (2.5).

Lemma 2.2. *The function*

$$R(z,t) = z^{1/3}\left\{\left[1 - \sqrt{1 - \frac{t^3}{27z^2}}\,\right]^{1/3} + \left[1 + \sqrt{1 - \frac{t^3}{27z^2}}\,\right]^{1/3}\right\}$$

is increasing and concave in z on $(0,\infty)$ and increasing in t on $\mathbb{R}$.

Proof. Since the function $Q(w)$ is the unique positive root of the equation (2.6) it follows that $Q^2(w) > 3w$ and therefore $Q^2(w) > w$ for any $w \geq 0$. Now, by differentiating (2.6) once we obtain

$$Q'(w)\left[Q^2(w) - w\right] = Q(w) > 0,$$

and consequently $Q'(w) > 0$ and also

$$Q(w) - 2wQ'(w) = \frac{Q(w)}{Q^2(w) - w}\left[Q^2(w) - 3w\right] > 0.$$

A second differentiation of (2.6) gives

$$Q''(w)\left[Q^2(w) - w\right] = 2\big[1 - Q(w)Q'(w)\big]Q'(w),$$

which can be written as

$$Q''(w)\left[Q^2(w) - w\right]^3 = -2wQ(w),$$

and therefore we conclude that $Q''(w) < 0$ for any $w > 0$, and $Q''(w) > 0$ if $w < 0$.

Now, from the definition of the function $R(z,t)$, observe that

$$\frac{\partial R}{\partial z} = \frac{1}{3z^{2/3}}\left[Q(w) - 2wQ'(w)\right]\Big|_{w=t/(3z^{2/3})} = \frac{1}{3z^{2/3}}\,\frac{Q(w)\left[Q^2(w) - 3w\right]}{Q^2(w) - w}\Bigg|_{w=t/(3z^{2/3})},$$

and also that

$$\frac{\partial^2 R}{\partial z^2} = \frac{2}{9z^{5/3}} \Big[-Q(w) + 3wQ'(w) + 2w^2 Q''(w) \Big]\Big|_{w=t/(3z^{2/3})}$$
$$= \begin{cases} \left.\dfrac{2Q''(w)}{9wz^{5/3}}\right|_{w=t/(3z^{2/3})}, & \text{if} \quad t>0, \\ -2^{4/3}/(9z^{5/3}), & \text{if} \quad t=0. \end{cases}$$

Consequently, $\dfrac{\partial R}{\partial z} > 0$, $\dfrac{\partial^2 R}{\partial z^2} < 0$ and $\dfrac{\partial R}{\partial t} > 0$. □

Lemma 2.3. *If $x_n \le B_n(t)$ for all $n \ge 0$, then $(\mathrm{T}x)_n \ge c_1 B_n(t)$ for all $n \ge 1$ and $(\mathrm{T}x)_0 \ge c_0$, where $c_0 = 0.68554389$ and $c_1 = 0.6379714$.*

Proof. Equation (2.1) easily gives

$$(\mathrm{T}x)_n^2 - t \ge \frac{n+1}{(\mathrm{T}x)_n + B_{n+1}(t)} + \frac{n}{(\mathrm{T}x)_n + B_{n-1}(t)} \quad \text{for} \quad n \ge 0. \tag{2.7}$$

With Lemma 2.2, we readily observe that for $z \in [0,\infty)$ the function $f(z) = 1/[y + R(z,t)]$ is such that

$$\frac{\partial f}{\partial z} = -\frac{1}{[y+R(z,t)]^2}\frac{\partial R}{\partial z} \le 0,$$

whilst

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{[y+R(z,t)]^3}\left(\frac{\partial R}{\partial z}\right)^2 - \frac{1}{[y+R(z,t)]^2}\frac{\partial^2 R}{\partial z^2} \ge 0.$$

Hence $f(z)$ is convex on $(0,\infty)$, so that

$$\lambda f(z_1) + (1-\lambda) f(z_2) \ge f(\lambda z_1 + (1-\lambda) z_2), \qquad \lambda \in [0,1],$$

whenever $0 < z_1 \le z_2 < \infty$. For $n \ge 1$ we choose $z_1 = n + \frac{3}{2}$, $z_2 = n - \frac{1}{2}$ and $\lambda = \frac{1}{2}$, then this gives for $y = (\mathrm{T}x)_n$

$$\frac{1}{(\mathrm{T}x)_n + R\left(n+\frac{3}{2},t\right)} + \frac{1}{(\mathrm{T}x)_n + R\left(n-\frac{1}{2},t\right)} \ge \frac{2}{(\mathrm{T}x)_n + R\left(n+\frac{1}{2},t\right)} \quad \text{for} \quad n \ge 1,$$

which easily leads to

$$\frac{n+1}{(\mathrm{T}x)_n + R\left(n+\frac{3}{2},t\right)} + \frac{n}{(\mathrm{T}x)_n + R\left(n-\frac{1}{2},t\right)} \ge \frac{2n}{(\mathrm{T}x)_n + R\left(n+\frac{1}{2},t\right)} \quad \text{for} \quad n \ge 1.$$

Combined with (2.7), and also because $B_n(t) = R(n+\frac{1}{2},t))$, this gives

$$(\mathrm{T}x)_n^2 - t \ge \frac{2n}{(\mathrm{T}x)_n + B_n(t)}, \quad \text{for} \quad n \ge 1.$$

and therefore

$$\left[(\mathrm{T}x)_n^2 - t\right]\left[(\mathrm{T}x)_n + B_n(t)\right] \ge 2n, \qquad \text{for} \quad n \ge 1.$$

With $c = (\mathrm{T}x)_n/B_n(t)$, the latter inequality

$$(c+1)\Big[c^2\,(\,B_n(t))^2 - t\Big]B_n(t) \geq 2n, \qquad \text{for} \quad n \geq 1,$$

which can be expressed as

$$(c+1)\Big[(c^2-1)B_n^3(t) + (2n+1)\Big] \geq 2n, \qquad \text{for} \quad n \geq 1,$$

because $tB_n(t) = B_n^3(t) - (2n+1)$. Since $B_n(t) = \left(n+\frac{1}{2}\right)^{1/3} Q(w_n(t))$, for $n \geq 0$, with $w_n(t) = t/[3(n+\frac{1}{2})^{2/3}]$, it follows

$$(c+1)\left\{(c^2-1)\Big[Q(w_n(t))\Big]^3 + 2\right\} \geq \frac{4n}{2n+1}, \qquad \text{for any} \quad n \geq 1. \tag{2.8}$$

For $t \geq 0$, the sequence $\left(w_n(t) = t/[3(n+\frac{1}{2})^{2/3}]\right)_{n\geq 1}$, is monotonically decreasing so that $0 < w_n(t) \leq w_1(t),\ n \geq 1$. Since $Q(z)$ is a positive, increasing function (with $\lim_{z\to-\infty} Q(z) = 0$, $\lim_{z\to+\infty} Q(z) = +\infty$), it follows that

$$Q(0) \leq Q(w_n(t)) \leq Q(w_1(t)),$$

for any $t \geq 0$ and $n \geq 1$, where $Q(0) = 2^{1/3}$ and

$$Q(w_1(t)) = \left(1 - \sqrt{1 - w_1^3(t)}\right)^{1/3} + \left(1 + \sqrt{1 - w_1^3(t)}\right)^{1/3}.$$

Since $0 < (\mathrm{T}x)_n \leq B_n(t)$ implies $0 < c < 1$ and therefore $(c+1)(c^2-1) < 0$, then

$$2c^2(c+1) \ \geq \ (c+1)\left\{(c^2-1)\left[Q(w_n(t))\right]^3 + 2\right\} \geq \frac{4n}{2n+1} \geq \frac{4}{3}.$$

Hence, c is such that $c^2(c+1) \geq \frac{2}{3}$. The function $2c^2(c+1)$ is monotone increasing on $[0,1]$, and this implies $c \geq c_1$, where c_1 is the positive root of $c^2(c+1) = \frac{2}{3}$, which is approximately $c_1 = 0.6379714$.

For $n = 0$ we use (2.7) to obtain

$$[(\mathrm{T}x)_0]^2 - t \geq 1/[(\mathrm{T}x)_0 + B_1(t)],$$

so that $[(\mathrm{T}x)_0 + B_1(t)]\,\{[(\mathrm{T}x)_0]^2 - t\} \geq 1$. With $d = (\mathrm{T}x)_0/B_1(t)$, then we have

$$(d+1)\left[d^2B_1^3(t) - tB_1(t)\right] \geq 1.$$

With the identity $tR\left(\frac{3}{2}, t\right) = B_1^3(t) - 3$, the latter inequality becomes

$$(d+1)\left[(d^2-1)B_1^3(t) + 3\right] \geq 1.$$

which, because of the fact that $B_1(t) \geq 3^{1/3}$ and $0 \leq d \leq 1$, implies

$$(d+1)\left[(d^2-1)3 + 3\right] \geq (d+1)\left[(d^2-1)B_1^3(t) + 3\right] \geq 1.$$

The function $d^2(d+1)$ is monotone increasing on $[0,1]$, and this implies $d \geq c_0$, where c_0 is the positive root of $d^2(d+1) = \frac{1}{3}$, which is approximately $c_0 = 0.47533$, so that $(\mathrm{T}x)_0 \geq 0.47533\ B_1(t) \geq 0.47533 \times 3^{1/3} = 0.685544$. $\square$

If we use the norm $\|x\| = \sup_{n\geq 0} |x_n|/B_n(t)$, with $B_n(t)$ as given in (2.4), then Lemma 2.1 implies that $\|\mathrm{T}x\| \leq 1$ whenever $(x_n)_{n\geq 0}$ is a positive sequence. Lemma 2.3 implies that $\|\mathrm{T}^2 x\| \geq c_1$. We are interested in the iterations $\mathrm{T}^k x$ for $k \to \infty$, so without loss of generality we may assume that $c_1 B_n(t) \leq x_n \leq B_n(t)$. The mapping T is then a mapping from positive sequences in the unit ball $\mathbf{B}_1 = \{(x_n)_{n\geq 0} : \|x\| \leq 1\}$ to positive sequences in the unit ball $\mathbf{B}_1$.

We will show that whenever $t \geq 0$, T is a contraction on the positive sequences for which $c_1 \leq \|x\| \leq 1$. Let x, y be positive sequences with $c_1 \leq \|x\|, \|y\| \leq 1$, then from (2.2)

$$\begin{aligned}\left|(\mathrm{T}x)_0 - (\mathrm{T}y)_0\right| &= \left|\sqrt{\frac{1}{(\mathrm{T}x)_0 + x_1} + t} - \sqrt{\frac{1}{(\mathrm{T}y)_0 + y_1} + t}\right| \\ &= \frac{1}{(\mathrm{T}x)_0 + (\mathrm{T}y)_0}\left|\frac{1}{(\mathrm{T}x)_0 + x_1} - \frac{1}{(\mathrm{T}y)_0 + y_1}\right|.\end{aligned}$$

From Lemma 2.3 it follows that $(\mathrm{T}x)_0 \geq c_0 B_1(t)$ and $(\mathrm{T}y)_0 \geq c_0 B_1(t)$, hence

$$\left|(\mathrm{T}x)_0 - (\mathrm{T}y)_0\right| \leq \frac{1}{2c_0 B_1(t)} \frac{|(\mathrm{T}y)_0 - (\mathrm{T}x)_0 + y_1 - x_1|}{[(\mathrm{T}x)_0 + x_1][(\mathrm{T}y)_0 + y_1]}.$$

Lemma 2.3 also gives $x_1 \geq c_1 B_1(t)$ and $y_1 \geq c_1 B_1(t)$ so that

$$\left|(\mathrm{T}x)_0 - (\mathrm{T}y)_0\right| \leq \frac{1}{2c_0(c_0+c_1)^2 B_1^3(t)}\Big[\left|(\mathrm{T}x)_0 - (\mathrm{T}y)_0\right| + |x_1 - y_1|\Big].$$

From this we find

$$B_0(t)\left[1 - \frac{1}{2c_0(c_0+c_1)^2 B_1^3(t)}\right]\frac{\left|(\mathrm{T}x)_0 - (\mathrm{T}y)_0\right|}{B_0(t)} \leq \frac{1}{2c_0(c_0+c_1)^2 B_1^2(t)}\frac{|x_1 - y_1|}{B_1(t)}.$$

For $t \geq 0$, the function $\left[1 - \frac{1}{2c_0(c_0+c_1)^2 B_1^3(t)}\right] > 0$, so that we can write

$$\frac{\left|(\mathrm{T}x)_0 - (\mathrm{T}y)_0\right|}{B_0(t)} \leq \frac{B_1(t)}{B_0(t)\left[2c_1(c_0+c_1)^2 B_1^3(t) - 1\right]}\frac{|x_1 - y_1|}{B_1(t)}$$

The function on the right is a decreasing function on $t \geq 0$, so that

$$\frac{B_1(t)}{B_0(t)\left[2c_0(c_0+c_1)^2 B_1^3(t) - 1\right]} \geq \frac{B_1(0)}{B_0(0)\left[2c_0(c_0+c_1)^2 B_0^3(t) - 1\right]} = 0.568967,$$

approximately. For this reason, we have

$$\frac{\left|(\mathrm{T}x)_0 - (\mathrm{T}y)_0\right|}{B_0(t)} \leq c_2 \frac{|x_1 - y_1|}{B_1(t)}, \qquad c_2 = 0.568967\ldots. \tag{2.9}$$

A similar calculation holds for $n \geq 1$: from (2.1) we find

$$\begin{aligned}
|(\mathrm{T}x)_n - (\mathrm{T}y)_n| &= \left| \sqrt{\frac{n+1}{(\mathrm{T}x)_n + x_{n+1}} + \frac{n}{(\mathrm{T}x)_n + x_{n-1}} + t} \right. \\
&\qquad \left. - \sqrt{\frac{n+1}{(\mathrm{T}y)_n + y_{n+1}} + \frac{n}{(\mathrm{T}y)_n + y_{n-1}} + t} \right| \\
&= \frac{1}{(\mathrm{T}x)_n + (\mathrm{T}y)_n} \left\{ \left| \frac{n+1}{(\mathrm{T}x)_n + x_{n+1}} - \frac{n+1}{(\mathrm{T}y)_n + y_{n+1}} \right| \right. \\
&\qquad \left. + \left| \frac{n}{(\mathrm{T}x)_n + x_{n-1}} - \frac{n}{(\mathrm{T}y)_n + y_{n-1}} \right| \right\}.
\end{aligned}$$

Lemma 2.3 gives $(\mathrm{T}x)_n \geq c_1 B_n(t)$ and $(\mathrm{T}y)_n \geq c_1 B_n(t)$ so that

$$\begin{aligned}
|(\mathrm{T}x)_n - (\mathrm{T}y)_n| \leq \frac{1}{2c_1 B_n(t)} &\left\{ \frac{(n+1)\big[|(\mathrm{T}y)_n - (\mathrm{T}x)_n| + |y_{n+1} - x_{n+1}|\big]}{[(\mathrm{T}x)_n + x_{n+1}][(\mathrm{T}y)_n + y_{n+1}]} \right. \\
&\qquad \left. + \frac{n\big[|(\mathrm{T}y)_n - (\mathrm{T}x)_n| + |y_{n-1} - x_{n-1}|\big]}{[(\mathrm{T}x)_n + x_{n-1}][(\mathrm{T}y)_n + y_{n-1}]} \right\}.
\end{aligned}$$

Since $x_{n+1}, y_{n+1} \geq c_1 B_{n+1}(t)$ and $x_{n-1}, y_{n-1} \geq c_1 B_{n-1}(t)$ we have (together with the lower bound from Lemma 2.3)

$$\begin{aligned}
|(\mathrm{T}x)_n - (\mathrm{T}y)_n| \leq& \frac{|(\mathrm{T}x)_n - (\mathrm{T}y)_n|}{2c_1^3 B_n(t)} \left\{ \frac{n+1}{[B_n(t) + B_{n+1}(t)]^2} + \frac{n}{[B_n(t) + B_{n-1}(t)]^2} \right\} \\
&+ \frac{\|x - y\|}{2c_1^3 B_n(t)} \left\{ \frac{(n+1)B_{n+1}(t)}{[B_n(t) + B_{n+1}(t)]^2} + \frac{nB_{n-1}(t)}{[B_n(t) + B_{n-1}(t)]^2} \right\},
\end{aligned}$$

which can be rearranged so that

$$\begin{aligned}
&\left\{ 1 - \frac{1}{2c_1^3 B_n(t)} \left\{ \frac{n+1}{[B_n(t) + B_{n+1}(t)]^2} + \frac{n}{[B_n(t) + B_{n-1}(t)]^2} \right\} \right\} \frac{|(\mathrm{T}x)_n - (\mathrm{T}y)_n|}{B_n(t)} \\
&\leq \quad \frac{\|x - y\|}{2c_1^3 B_n(t)^2} \left\{ \frac{(n+1)B_{n+1}(t)}{[B_n(t) + B_{n+1}(t)]^2} + \frac{nB_{n-1}(t)}{[B_n(t) + B_{n-1}(t)]^2} \right\}.
\end{aligned}$$

The function on the left

$$f(n,t) = 1 - \frac{1}{2c_1^3 B_n(t)} \left\{ \frac{n+1}{[B_n(t) + B_{n+1}(t)]^2} + \frac{n}{[B_n(t) + B_{n-1}(t)]^2} \right\}$$

is bounded from below by $f(n,0)$ since $B_n(t) \geq (2n+1)^{1/3}$ for any $n \geq 0$. In addition, $f(n,0)$ is an increasing function of n and therefore $f(n,0)$ is bounded from below by its value at $n = 1$, which is approximately $f(1,0) = 0.507422$. Since $f(n,t) \geq 0.507422 > 0$, we can write

$$\frac{|(\mathrm{T}x)_n - (\mathrm{T}y)_n|}{B_n(t)} \leq \frac{\dfrac{(n+1)B_{n+1}(t)}{[B_n(t) + B_{n+1}(t)]^2} + \dfrac{nB_{n-1}(t)^2}{[B_n(t) + B_{n-1}(t)]^2}}{2c_1^3 B_n(t) - \left\{ \dfrac{n+1}{[B_n(t) + B_{n+1}(t)]^2} + \dfrac{n}{[B_n(t) + B_{n-1}(t)]^2} \right\}} \|x - y\|.$$

We observe that for each $t \geq 0$, the expression on the right is an increasing sequence for $n \geq 1$, therefore it is bounded from above by the values as $n \to \infty$, which is approximately $c_3 = 1/(8c_1^3 - 1) = 0.928273$. Consequently, for any $n \geq 1$

$$\frac{|(\mathrm{T}x)_n - (\mathrm{T}y)_n|}{B_n(t)} \leq c_3\|x - y\|, \qquad c_3 = 0.928273\ldots. \tag{2.10}$$

Combining (2.9) and (2.10) then gives

$$\|\mathrm{T}x - \mathrm{T}y\| \leq 0.928273\|x - y\|,$$

which shows that T is a contraction. Since the unit ball with the norm $\|\cdot\|$ is complete, one can use Banach's fixed point theorem to conclude that T has a unique fixed point b for which $\mathrm{T}b = b$. The sequence $b = (b_n)_{n\geq 0}$ is positive and it is a solution of the alternative discrete Painlevé equation (1.4). The contraction factor can be improved by improving the upper and the lower bounds in Lemmas 2.1 and 2.3. The lower bound in Lemma 2.3 can be used to get a better upper bound in Lemma 2.1, which in turn can be used to improve the lower bound in Lemma 2.3.

We have not yet shown that $b_0(t) = -\operatorname{Ai}'(t)/\operatorname{Ai}(t)$; see Corollary 5.2.

3 Behaviour of the unique positive solution

The unique positive solution of (1.4) necessarily satisfies $b_n(t) > \sqrt{t}$, which is an immediate consequence of (1.4). Furthermore, Lemma 2.1 implies $\sqrt{t} < b_n(t) \leq B_n(t)$, for any $n \geq 0$ and $t \geq 0$. This is illustrated in Figure 3.1 for $n = 5$ and $n = 10$. Moreover, recalling the

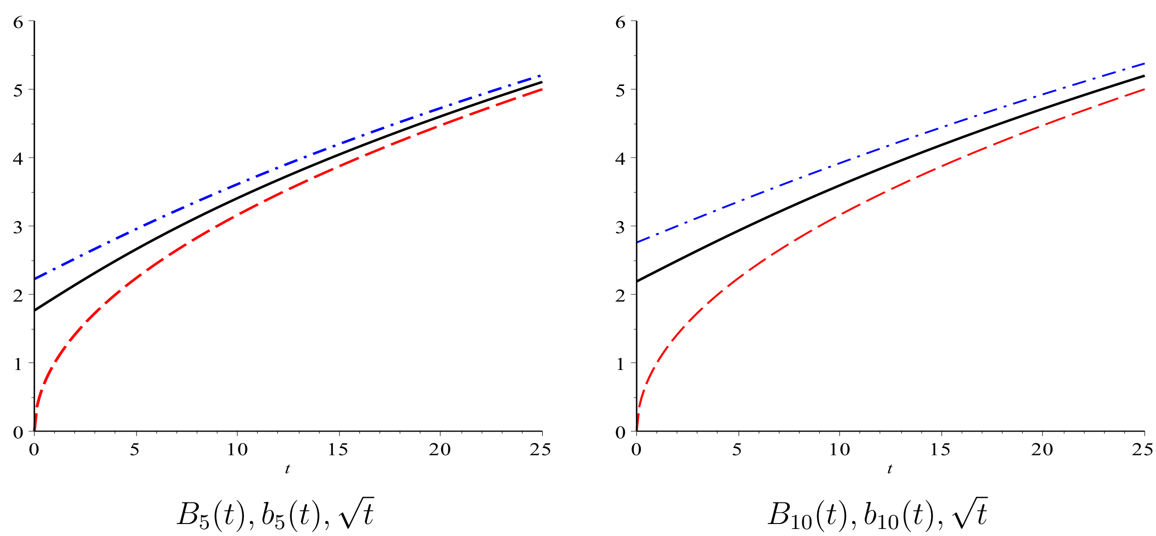

Figure 3.1: Plots of $B_n(t)$ (dot-dash line), $b_n(t)$ (solid line) and $\sqrt{t}$ (dashed line) for $n = 5$ and $n = 10$.

definition of $B_n(t)$ given by (2.4), we observe that $\lim_{t\to\infty} B_n(t)/\sqrt{t} = 1$, and consequently,

$$\lim_{t\to\infty} b_n(t)/\sqrt{t} = 1. \tag{3.1}$$

Remark 3.1. Observe that for any $t \geq 0$ and any integer $n \geq 0$, the upper bound $B_n(t)$ for $b_n(t)$ given in Lemma 2.1 satisfies $\sqrt{t} < B_n(t) < \sqrt{t + (2n+1)^{2/3}}$. For values of $t > 3\left(n + \frac{1}{2}\right)^{2/3}$, $B_n(t)$ can be written as

$$B_n(t) = \tfrac{2}{3}\sqrt{3\,t}\cos\left\{\tfrac{1}{3}\arccos\left[3^{3/2}(n+\tfrac{1}{2})/t^{3/2}\right]\right\},$$

for any $n \geq 0$. Since $0 \leq \frac{1}{3}\arccos\left[3^{3/2}(n+\frac{1}{2})/t^{3/2}\right] < \frac{1}{6}\pi$ for $t > 3\left(n+\frac{1}{2}\right)^{2/3}$, it follows that $\sqrt{t} < B_n(t) \leq \frac{2}{3}\sqrt{3\,t}$.

Remark 3.2. The Bäcklund transformations pair

$$b_n(t) + b_{n+1}(t) = \frac{2(n+1)}{b_n^2(t) - t + b_n'(t)}, \tag{3.2a}$$

$$b_n(t) + b_{n-1}(t) = \frac{2n}{b_n^2(t) - t - b_n'(t)}, \tag{3.2b}$$

relate solutions of the equation (1.4). In fact, (1.4) can be obtained by eliminating $b_n'(t)$ between (3.2a) and (3.2b). Hence,

$$b_n'(t) = \frac{n+1}{b_n(t) + b_{n+1}(t)} - \frac{n}{b_n(t) + b_{n-1}(t)},$$

which, recalling (1.1b), corresponds to

$$b_n'(t) = a_{n+1}(t) - a_n(t). \tag{3.3}$$

Moreover, (1.1a) and the latter equality yield

$$b_{n+1}^2(t) - b_n^2(t) = b_{n+1}'(t) + b_n'(t). \tag{3.4}$$

The identities (3.3) and (3.4) allow us to conclude the following result.

Lemma 3.1. *For each integer $n \geq 0$, $b_n'(t) > 0$ for all $t \geq 0$ if and only if $a_{n+1}(t) > a_n(t)$, and this implies $b_{n+1}(t) > b_n(t)$ and also $a_n'(t) < 0$.*

Numerically it is evident that $b_n(t)$ is a strictly increasing function of t. However, it remains an open problem to prove this (analytically).

Conjecture 3.2. *For any $t \geq 0$, $0 < b_n'(t) < 1/(2\sqrt{t})$.*

Notwithstanding this, we can still prove the result for values of t larger than $2^{-2/3}n^{4/3}$.

Lemma 3.3. *For each integer $n \geq 0$, we have $b_n'(t) > 0$ for any $t \geq \left(\frac{(2n+3)n^3}{(2n+1)(n+1)}\right)^{2/3}$.*

Proof. Since $a_0 = 0$, then (3.3) readily implies $b_0'(t) = a_1(t) > 0$, for any $t \geq 0$. Based on (3.3) together with the inequalities

$$\frac{n}{2\sqrt{t}} < a_n < \frac{n}{B_n(t) + B_{n-1}(t)},$$

which are valid for any $n \geq 1$ and $t \geq 0$, it follows that

$$b_n'(t) > \frac{n+1}{B_{n+1}(t) + B_n(t)} - \frac{n}{2\sqrt{t}}.$$

Since $B_n(t)$ is an increasing function of both n and t, bounded from below by $\sqrt{t}$ and $\lim_{t\to+\infty} \frac{B_n(t)}{\sqrt{t}} = 1$, it follows that for each $n \geq 1$ there exists t_n^* such that for any $t > t_n^*$, we have $B_n(t) + B_{n+1}(t) \leq \left(1 + \frac{1}{n}\right) 2\sqrt{t}$, which implies $b_n'(t) > 0$.

Indeed, let t_n^* be such that $B_{n+1}(t_n^*) = \frac{n+1}{n}\sqrt{t_n^*}$. As $B_n(t)$ is by definition the positive solution of $y(y^2 - t) = 2n + 1$, it readily follows that t_n^* must be such that

$$\frac{n+1}{n^3}\left[(n+1)^2 - n^2\right]\left(t_n^*\right)^{3/2} = 2n + 3,$$

and hence $t_n^* = \left[\frac{(2n+3)n^3}{(2n+1)(n+1)}\right]^{2/3}$. Then for $t > t_n^*$, $B_n(t) + B_{n+1}(t) \leq 2B_{n+1}(t)$, for each $n \geq 0$. $\square$

In the same manner we can show the existence of t_n^+ such that for any $t > t_n^+$, we have $b_n'(t) < 1/(2\sqrt{t})$.

Remark 3.3. The asymptotic behaviour of b_n for large t in (3.1) allows us to conclude from (1.1b) and (3.3), respectively, that $\lim_{t\to+\infty}\left(a_n(t)\,2\sqrt{t}/n\right) = 1$ and $\lim_{t\to+\infty}\left(b_n'(t)\,2\sqrt{t}\right) = 1$.

Remark 3.4. As $b_n(t) > \sqrt{t}$ for any $n \geq 0$, then (3.2a) implies

$$0 < b_n(t) + b_{n+1}(t) = \frac{2(n+1)}{b_n^2(t) - t + b_n'(t)}$$

whereas (3.2b) implies that

$$0 < b_n(t) + b_{n-1}(t) = \frac{2n}{b_n^2(t) - t - b_n'(t)}.$$

Hence, we have

$$\left[b_n^2(t) - t + b_n'(t)\right]\left[b_n^2(t) - t - b_n'(t)\right] > 0,$$

i.e., $-(b_n^2(t) - t) < b_n'(t) < b_n^2(t) - t$.

4 Relation with the second Painlevé equation

Suppose we denote solutions of $\mathrm{P_{II}}$ (1.3) by $w(z;\alpha)$, then the three solutions $w(z;\alpha)$ and $w(z;\alpha\pm 1)$ are related by

$$\frac{\alpha+\frac{1}{2}}{w(z;\alpha+1)+w(z;\alpha)}+\frac{\alpha-\frac{1}{2}}{w(z;\alpha)+w(z;\alpha-1)}+2w^2(z;\alpha)+z=0, \tag{4.1}$$

see [18, Eq. 32.7.5 on p. 730] and [7, Eq. (1.21)]. The latter is equivalent to the difference equation (1.4). If we set $\alpha=n+\frac{1}{2}$ then $b_n(t)=-2^{1/3}w(-2^{1/3}t;n+\frac{1}{2})$. It is known that for $\alpha=n+\frac{1}{2}$ then $\mathrm{P_{II}}$ (1.3) has solutions in terms of Airy functions (see [18, 32.10(ii) on p. 735] or [4, §7.1 on p. 373]). The simplest "Airy solution" is

$$w(z;\tfrac{1}{2})=-\frac{\mathrm{d}}{\mathrm{d}z}\ln\phi(z)=-\frac{\phi'(z)}{\phi(z)},$$

where

$$\phi(z)=C_1\,\mathrm{Ai}(-2^{-1/3}z)+C_2\,\mathrm{Bi}(-2^{-1/3}z), \tag{4.2}$$

with $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$ the Airy functions and C_1 and C_2 arbitrary constants. Observe that the solution $w(z;\frac{1}{2})$ only depends on the ratio C_1/C_2. We now have that

$$y_0(t)=-2^{-1/3}w(-2^{1/3}t;\tfrac{1}{2})=2^{-1/3}\frac{\phi'(-2^{1/3}t)}{\phi(-2^{1/3}t)},$$

and since $\phi(-2^{-1/3}t)=C_1\,\mathrm{Ai}(t)+C_2\,\mathrm{Bi}(t)$, then we find

$$y_0(t)=-\frac{C_1\,\mathrm{Ai}'(t)+C_2\,\mathrm{Bi}'(t)}{C_1\,\mathrm{Ai}(t)+C_2\,\mathrm{Bi}(t)}. \tag{4.3}$$

For $t=0$ this gives

$$b_0(0)=-\frac{C_1\,\mathrm{Ai}'(0)+C_2\,\mathrm{Bi}'(0)}{C_1\,\mathrm{Ai}(0)+C_2\,\mathrm{Bi}(0)}=\frac{3[\Gamma(\frac{2}{3})]^2}{2\pi}\left(\frac{C_1-\sqrt{3}\,C_2}{C_1+\sqrt{3}\,C_2}\right),$$

where we used the initial values

$$\mathrm{Ai}(0)=\frac{1}{3^{2/3}\Gamma(\frac{2}{3})},\quad \mathrm{Bi}(0)=\frac{1}{3^{1/6}\Gamma(\frac{2}{3})},\quad \mathrm{Ai}'(0)=-\frac{3^{1/6}\Gamma(\frac{2}{3})}{2\pi},\quad \mathrm{Bi}'(0)=\frac{3^{2/3}\Gamma(\frac{1}{3})}{2\pi},$$

see [18, 9.2(ii) on p. 194]. Thus choosing $C_2=0$ gives the initial value mentioned in Theorems 1.1 and 1.2 for $t=0$.

More generally, we have the solution

$$w(z;n+\tfrac{1}{2})=\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\Theta_n(z)}{\Theta_{n+1}(z)},\qquad \Theta_n(z)=\det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}}\phi(z)\right]_{j,k=0}^{n-1}, \tag{4.4}$$

for $n\geq 1$, with $\phi(z)$ as given by (4.2), cf. [4, 10, 17].

Consider the Bäcklund transformations

$$y_{n+1} = -y_n + \frac{2(n+1)}{y_n^2 + y_n' - t}, \tag{4.5a}$$

$$y_{n-1} = -y_n + \frac{2n}{y_n^2 - y_n' - t}, \tag{4.5b}$$

where $' \equiv d/dt$. Eliminating y_n' yields the recurrence relation

$$\frac{n+1}{y_{n+1} + y_n} + \frac{n}{y_n + y_{n-1}} = y_n^2 - t, \tag{4.6}$$

which is alt-dP$_\mathrm{I}$ (1.4), whilst letting $n \to n+1$ in (4.5b) and then substituting (4.5a) yields

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n^3 - 2ty_n - 2n - 1, \tag{4.7}$$

which is (1.2) with $\alpha = n + \frac{1}{2}$, and so is equivalent to P$_\mathrm{II}$ (1.3) with $\alpha = n + \frac{1}{2}$.

Lemma 4.1. *Suppose x_n and y_n satisfy the discrete system*

$$x_n + x_{n+1} = y_n^2 - t, \tag{4.8a}$$

$$x_n(y_n + y_{n-1}) = n, \tag{4.8b}$$

and y_n satisfies (4.5)*. Then x_n and y_n satisfy the system*

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = -2x_n y_n + n, \tag{4.9a}$$

$$\frac{\mathrm{d}y_n}{\mathrm{d}t} = y_n^2 - 2x_n - t, \tag{4.9b}$$

and x_n satisfies

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n}. \tag{4.10}$$

Proof. From (4.8b) and (4.5b) we have

$$\frac{2n}{y_n + y_{n-1}} = 2x_n = y_n^2 - \frac{\mathrm{d}y_n}{\mathrm{d}t} - t,$$

from which we obtain equation (4.9b). Then differentiating (4.9b) gives

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n \frac{\mathrm{d}y_n}{\mathrm{d}t} - 2\frac{\mathrm{d}x_n}{\mathrm{d}t} - 1.$$

Substituting for the derivatives of y_n using (4.9b) and (4.7) yields equation (4.9a). Finally solving (4.9a) for y_n and substituting in (4.9b) shows that x_n satisfies equation (4.10), as required. □

We remark that making the transformation $x_n(t) = -2^{-1/3}v(z)$, with $z = -2^{1/3}t$, in equation (4.10) yields

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = \frac{1}{2v}\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - 2v^2 - zv - \frac{n^2}{2v}, \tag{4.11}$$

which is known as P_{34}, as it's equivalent to equation XXXIV of Chapter 14 in [12].

The "Airy-type" solutions of (4.7) and (4.10) respectively have the form

$$y_n(t;\theta) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t;\theta)}{\tau_{n+1}(t;\theta)}, \qquad x_n(t;\theta) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\tau_n(t;\theta), \tag{4.12}$$

where

$$\tau_n(t;\theta) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\varphi(t;\theta)\right]_{j,k=0}^{n-1}, \tag{4.13}$$

with $\tau_0(t;\theta) = 1$ and $\varphi(t;\theta) = \cos(\theta)\,\mathrm{Ai}(t) + \sin(\theta)\,\mathrm{Bi}(t)$, with $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$ the Airy functions and $\theta \in [0,\pi)$ a parameter; we have set $C_1 = \cos(\theta)$ and $C_2 = \sin(\theta)$ to reflect that the solution only depends on the ratio of the constants. The "Airy-type" solutions (4.12) are derived from the "Airy-type" solutions of $\mathrm{P_{II}}$ (1.3) given in [4, 10, 17].

The simplest non-trivial "Airy-type" solutions of (4.7) and (4.10) respectively have the form

$$y_0(t;\theta) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln\varphi(t;\theta) = -\frac{\cos(\theta)\,\mathrm{Ai}'(t) + \sin(\theta)\,\mathrm{Bi}'(t)}{\cos(\theta)\,\mathrm{Ai}(t) + \sin(\theta)\,\mathrm{Bi}(t)}, \tag{4.14a}$$

$$x_1(t;\theta) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\varphi(t;\theta) = \left[\frac{\cos(\theta)\,\mathrm{Ai}'(t) + \sin(\theta)\,\mathrm{Bi}'(t)}{\cos(\theta)\,\mathrm{Ai}(t) + \sin(\theta)\,\mathrm{Bi}(t)}\right]^2 - t, \tag{4.14b}$$

recall that $x_0(t;\theta) = 0$. The structure of the solutions (4.14a) and (4.14b) depends critically on whether the parameter θ is zero or not which is shown in Lemmas 4.2 and 4.3 below. As $\mathrm{Ai}(t)$ decays exponentially as $t \to \infty$, whereas $\mathrm{Bi}(t)$ increases exponentially as $t \to \infty$, so if $\theta \neq 0$ then $\mathrm{Bi}(t)$ will dominate for large positive t.

If we seek a solution of (4.7) with $y_n(t) \sim c\,t^{1/2}$, as $t \to \infty$, then necessarily $c = \pm 1$. Hence the following asymptotic series are easily derived

$$y_n^+(t) = t^{1/2} + \frac{2n+1}{4\,t} - \frac{12n^2+12n+5}{32\,t^{5/2}} + \mathcal{O}\big(t^{-4}\big), \tag{4.15a}$$

$$y_n^-(t) = -t^{1/2} + \frac{2n+1}{4\,t} + \frac{12n^2+12n+5}{32\,t^{5/2}} + \mathcal{O}\big(t^{-4}\big). \tag{4.15b}$$

It should be noted that there are no arbitrary constants in these asymptotic expansions, they occur in exponentially small terms.

Lemma 4.2. *If $y_0(t;\theta)$ is given by* (4.14a), *then*

$$y_0(t;\theta) = \begin{cases} t^{1/2} + \mathcal{O}\big(t^{-1}\big), & \text{if} \quad \theta = 0, \\ -t^{1/2} + \mathcal{O}\big(t^{-1}\big), & \text{if} \quad \theta \neq 0. \end{cases} \tag{4.16}$$

Proof. Using the known asymptotics of $\operatorname{Ai}(t)$ and $\operatorname{Bi}(t)$

$$
\begin{aligned}
\operatorname{Ai}(t) &= \frac{\mathrm{e}^{-\zeta}}{2\sqrt{\pi}\,t^{1/4}}\left\{1-\frac{5}{48\,t^{3/2}}+\frac{385}{4608\,t^{3}}-\frac{85085}{663552\,t^{9/2}}+\mathcal{O}\big(t^{-6}\big)\right\}\\
\operatorname{Ai}'(t) &= -\frac{t^{1/4}\,\mathrm{e}^{-\zeta}}{2\sqrt{\pi}}\left\{1+\frac{7}{48\,t^{3/2}}+\frac{455}{4608\,t^{3}}+\frac{95095}{663552\,t^{9/2}}+\mathcal{O}\big(t^{-6}\big)\right\}\\
\operatorname{Bi}(t) &= \frac{\mathrm{e}^{\zeta}}{\sqrt{\pi}\,t^{1/4}}\left\{1+\frac{5}{48\,t^{3/2}}+\frac{385}{4608\,t^{3}}+\frac{85085}{663552\,t^{9/2}}+\mathcal{O}\big(t^{-6}\big)\right\}\\
\operatorname{Bi}'(t) &= \frac{t^{1/4}\,\mathrm{e}^{\zeta}}{\sqrt{\pi}}\left\{1-\frac{7}{48\,t^{3/2}}-\frac{455}{4608\,t^{3}}-\frac{95095}{663552\,t^{9/2}}+\mathcal{O}\big(t^{-6}\big)\right\}
\end{aligned}
$$

with $\zeta = \frac{2}{3}t^{3/2}$, see [18, 9.7(ii) on p. 198], then if $\theta = 0$, as $t \to \infty$

$$
\begin{aligned}
y_0(t;0) = -\frac{\operatorname{Ai}'(t)}{\operatorname{Ai}(t)} &= t^{1/2}\frac{\left\{1+\dfrac{7}{48\,t^{3/2}}+\dfrac{455}{4608\,t^{3}}+\dfrac{95095}{663552\,t^{9/2}}+\mathcal{O}\big(t^{-6}\big)\right\}}{\left\{1-\dfrac{5}{48\,t^{3/2}}+\dfrac{385}{4608\,t^{3}}-\dfrac{85085}{663552\,t^{9/2}}+\mathcal{O}\big(t^{-6}\big)\right\}}\\
&= t^{1/2}+\frac{1}{4\,t}-\frac{5}{32\,t^{5/2}}+\frac{15}{64\,t^{4}}+\mathcal{O}\big(t^{-11/2}\big)
\end{aligned}
$$

whilst if $\theta \neq 0$, as $t \to \infty$

$$
\begin{aligned}
y_0(t;\theta) &= -\frac{\cos(\theta)\operatorname{Ai}'(t)+\sin(\theta)\operatorname{Bi}'(t)}{\cos(\theta)\operatorname{Ai}(t)+\sin(\theta)\operatorname{Bi}(t)}\\
&= -t^{1/2}\frac{\frac{1}{2}\cos(\theta)\,\mathrm{e}^{-\zeta}\left\{1+\dfrac{7}{48\,t^{3/2}}+\mathcal{O}\big(t^{-3}\big)\right\}+\sin(\theta)\,\mathrm{e}^{\zeta}\left\{1-\dfrac{7}{48\,t^{3/2}}+\mathcal{O}\big(t^{-3}\big)\right\}}{\frac{1}{2}\cos(\theta)\,\mathrm{e}^{-\zeta}\left\{1-\dfrac{5}{48\,t^{3/2}}+\mathcal{O}\big(t^{-3}\big)\right\}+\sin(\theta)\,\mathrm{e}^{\zeta}\left\{1+\dfrac{5}{48\,t^{3/2}}+\mathcal{O}\big(t^{-3}\big)\right\}}\\
&= -t^{1/2}\frac{\left\{1-\dfrac{7}{48\,t^{3/2}}-\dfrac{455}{4608\,t^{3}}-\dfrac{95095}{663552\,t^{9/2}}+\mathcal{O}\big(t^{-6}\big)\right\}}{\left\{1+\dfrac{5}{48\,t^{3/2}}+\dfrac{385}{4608\,t^{3}}+\dfrac{85085}{663552\,t^{9/2}}+\mathcal{O}\big(t^{-6}\big)\right\}}+\mathcal{O}\left(t^{1/2}\,\mathrm{e}^{-2\zeta}\right)\\
&= -t^{1/2}+\frac{1}{4\,t}+\frac{5}{32\,t^{5/2}}+\frac{15}{64\,t^{4}}+\mathcal{O}\big(t^{-11/2}\big)
\end{aligned}
$$

□

Plots of $y_0(t;\theta)$ and $x_1(t;\theta)$ for various values of the parameter θ are given in Figure 4.1. We note that

$$
y_0(0;\theta) = -\frac{\cos(\theta)\operatorname{Ai}'(0)+\sin(\theta)\operatorname{Bi}'(0)}{\cos(\theta)\operatorname{Ai}(0)+\sin(\theta)\operatorname{Bi}(0)} = \frac{3^{5/6}}{2\pi}\left(\frac{1-\sqrt{3}\,\tan\theta}{1+\sqrt{3}\,\tan\theta}\right)\left[\Gamma(\tfrac{2}{3})\right]^2,
$$

which is a decreasing function for $0 \le \theta \le \frac{1}{2}\pi$.

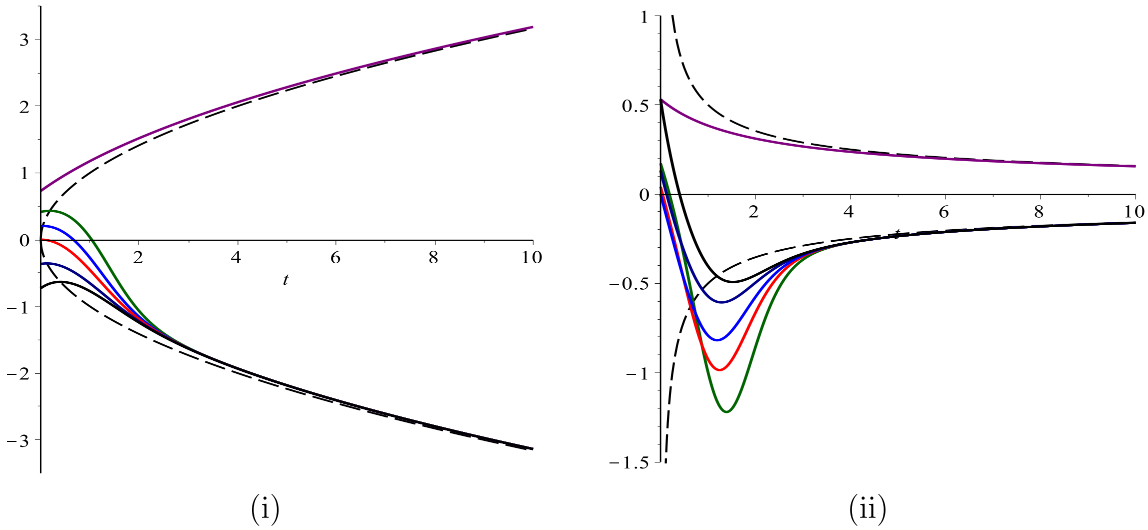

Figure 4.1: (i) Plots of $y_0(t;\theta)$ for $\theta = 0$ (upper curve), $\theta = \frac{1}{20}\pi, \frac{1}{10}\pi, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi$ (lower curves, with $\theta = \frac{1}{20}\pi$ the highest and $\theta = \frac{1}{2}\pi$ the lowest), and the curves $y = \pm\sqrt{t}$ (dashed lines).(ii) Plots of $x_1(t;\theta)$ for $\theta = 0$ (upper curve), $\theta = \frac{1}{20}\pi, \frac{1}{10}\pi, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi$ (lower curves, with $\theta = \frac{1}{20}\pi$ the lowest and $\theta = \frac{1}{2}\pi$ the highest), and the curves $x = \pm 1/(2\sqrt{t})$ (dashed lines).

An analogous situation arises for the solution $y_n(t;\theta)$ given in (4.12), i.e.

$$y_n(t;\theta) = \begin{cases} t^{1/2} + \mathcal{O}\big(t^{-1}\big), & \text{if} \quad \theta = 0, \\ -t^{1/2} + \mathcal{O}\big(t^{-1}\big), & \text{if} \quad \theta \neq 0 \end{cases}$$

(see Lemma 5.1 below). We remark that Fornberg and Weideman [9, Fig. 3 on p. 991] plot the locations of the poles for the solution $w(z;\frac{5}{2})$ of $\mathrm{P_{II}}$ (1.3), which is equivalent to $y_2(t;\theta)$, for various choices of θ. These plots show that the pole structure of the solutions is significantly different in the case when $\theta = 0$ compared to the case when $\theta \neq 0$. A study of “Airy-type” solutions of $\mathrm{P_{II}}$ (1.3) and $\mathrm{P_{34}}$ (4.11), which are equivalent to $y_n(t;\theta)$ and $x_n(t;\theta)$, for various choices of θ is given in [5].

Lemma 4.3. *If* $x_1(t;\theta)$ *is given by* (4.14b), *then*

$$x_1(t;\theta) = \begin{cases} \frac{1}{2}\, t^{-1/2} + \mathcal{O}\big(t^{-2}\big), & \text{if} \quad \theta = 0, \\ -\frac{1}{2}\, t^{-1/2} + \mathcal{O}\big(t^{-2}\big), & \text{if} \quad \theta \neq 0. \end{cases} \tag{4.17}$$

Proof. The proof is very similar to that for Lemma 4.2 so is left to the reader. $\square$

5 Airy solutions of alternative discrete Painlevé I

Now we consider the case when $\theta = 0$, i.e.

$$a_n(t) = x_n(t;0) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t), \qquad b_n(t) = y_n(t;0) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_n(t)}{\Delta_{n+1}(t)}, \tag{5.1}$$

where

$$\Delta_n(t) = \tau_n(t;0) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\,\mathrm{Ai}(t)\right]_{j,k=0}^{n-1}, \tag{5.2}$$

with $\Delta_0(t) = 1$. We remark that $\Delta_n(t)$ given by (5.2) arises in random matrix theory, in connection with the Gaussian Unitary Ensemble (GUE) in the soft-edge scaling limit, see e.g. [10, p. 393]. Further, for $n \geq 1$, $\Delta_n(t)$ has the multiple integral representation

$$\Delta_n(t) = \frac{(-1)^n}{(2\pi\mathrm{i})^n}\int_{\mathcal{C}}\cdots\int_{\mathcal{C}}\prod_{j=1}^{n}\exp\left(\tfrac{1}{3}x_j^3 - tx_j\right)\prod_{1\le k<\ell\le n}(x_k - x_\ell)^2\,\mathrm{d}x_1\,\ldots\,\mathrm{d}x_n,$$

in which $\mathcal{C}$ is the standard Airy contour from $\infty\mathrm{e}^{-\pi\mathrm{i}/3}$ to $\infty\mathrm{e}^{\pi\mathrm{i}/3}$ [10, p. 393].

Lemma 5.1. *If $a_n(t)$ and $b_n(t)$ satisfies the recurrence relation* (1.1) *with*

$$a_0(t) = 0, \qquad b_0(t) = -\mathrm{Ai}'(t)/\mathrm{Ai}(t), \tag{5.3}$$

where $\mathrm{Ai}(t)$ *is the Airy function, then as $t\to\infty$*

$$a_n(t) = \frac{n}{2\,t^{1/2}} + \mathcal{O}\big(t^{-2}\big), \qquad \text{for} \quad n \geq 1, \tag{5.4a}$$

$$b_n(t) = t^{1/2} + \mathcal{O}\big(t^{-1}\big), \qquad \text{for} \quad n \geq 0. \tag{5.4b}$$

Proof. We shall first prove (5.4b) by induction. If $b_n(t) = y_n(t;0)$ satisfies the recurrence relation (4.6), which is a consequence of (1.1), then $b_n(t)$ also satisfies the differential-difference system (4.5) and the differential equation (4.7). We shall use the Bäcklund transformation (4.5a).

Clearly (5.4b) is satisfied for $n = 0$ from Lemma 4.2. Now we assume as the inductive hypothesis $b_n(t) = t^{1/2} + \mathcal{O}\big(t^{-1}\big)$, so we know from (4.15a) that

$$b_n(t) = t^{1/2} + \frac{2n+1}{4\,t} - \frac{12n^2+12n+5}{32\,t^{5/2}} + \frac{(2n+1)(16n^2+16n+15)}{64\,t^4} + \mathcal{O}\big(t^{-11/2}\big),$$

and hence it is easily shown that

$$\begin{aligned} b_n^2(t) &= t + \frac{2n+1}{2\,t^{1/2}} - \frac{2n^2+2n+1}{4t^2} + \frac{5(2n+1)(4\,n^2+4n+5)}{64\,t^{7/2}} + \mathcal{O}\big(t^{-5}\big)\\ b_n'(t) &= \frac{1}{2\,t^{1/2}} - \frac{2n+1}{4\,t^2} + \frac{5(12\,n^2+12n+5)}{64\,t^{7/2}} + \mathcal{O}\big(t^{-5}\big). \end{aligned}$$

Therefore

$$b_n^2(t) + b_n'(t) - t = \frac{n+1}{t^{1/2}}\left\{1 - \frac{n+1}{2\,t^{3/2}} + \frac{5(4n^2+8n+5)}{32\,t^3} + \mathcal{O}\big(t^{-9/2}\big)\right\},$$

then

$$\frac{2(n+1)}{b_n^2(t)+b_n'(t)-t} = 2\,t^{1/2} + \frac{n+1}{t} - \frac{12n^2+24n+17}{16\,t^{5/2}} + \mathcal{O}\big(t^{-4}\big),$$

and so from (4.5a)

$$\begin{aligned} b_{n+1}(t) &= -b_n(t) + \frac{2(n+1)}{b_n^2(t)+b_n'(t)-t} \\ &= t^{1/2} + \frac{2n+3}{4\,t} - \frac{12n^2+36n+29}{32\,t^{5/2}} + \mathcal{O}\big(t^{-4}\big), \end{aligned}$$

which is (4.15a) with $n \to n+1$. Hence the result (5.4b) follows by induction. The result (5.4a) then follows immediately from (1.1b). $\square$

Corollary 5.2. *The positive solution $b_n(t)$ in Theorems 1.1 and 1.2 corresponds to the initial value $b_0(t) = -\operatorname{Ai}'(t)/\operatorname{Ai}(t)$.*

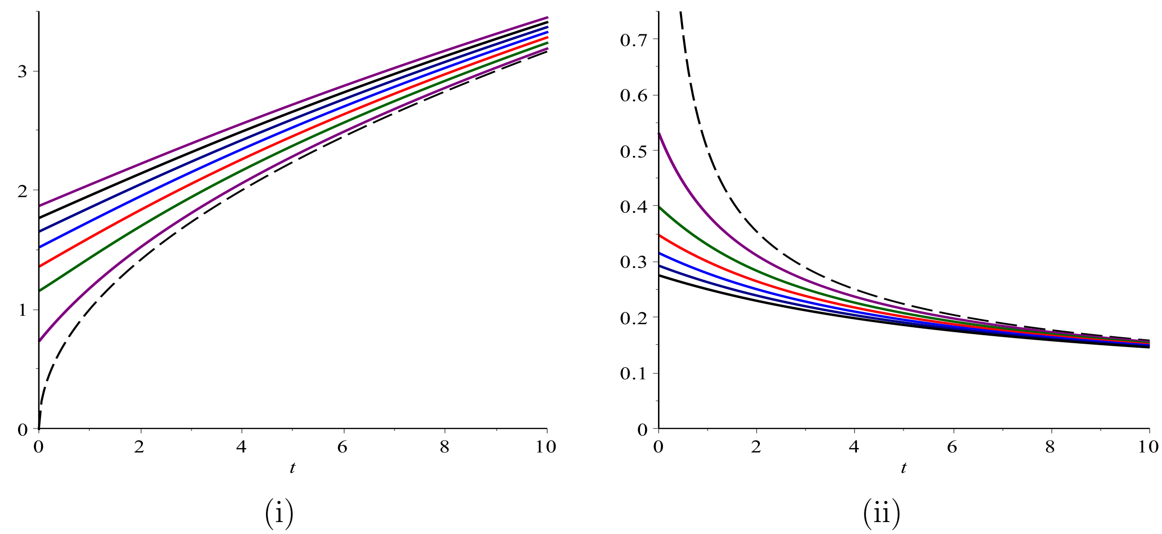

Figure 5.1: (i) Plots of $b_n(t) = y_n(t;0)$ for $n = 0, 1, \ldots, 6$, and the curve $y = \sqrt{t}$ (dashed line). (ii) Plots of $a_n(t)/n = x_n(t;0)/n$ for $n = 1, 2, \ldots, 6$, and the the curve $x = 1/(2\sqrt{t})$ (dashed line).

Plots of $b_n(t) = y_n(t;0)$, for $n = 0, 1, \ldots, 6$, and plots of $a_n(t)/n = x_n(t;0)/n$, for $n = 1, 2, \ldots, 6$, are given in Figure 5.1. These plots suggest the following conjecture, which corroborate Conjecture 3.2 and Lemma 3.3.

Conjecture 5.3. *If $0 < t_1 < t_2$ then*

$$b_n(t_1) < b_n(t_2), \qquad a_n(t_1) > a_n(t_2), \tag{5.5}$$

i.e. $b_n(t)$ is monotonically increasing and $a_n(t)$ is monotonically decreasing for $t > 0$. For fixed t with $t > 0$ then

$$\sqrt{t} < b_n(t) < b_{n+1}(t), \qquad \frac{1}{2\sqrt{t}} > \frac{a_n(t)}{n} > \frac{a_{n+1}(t)}{n+1} > 0. \tag{5.6a}$$

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References

[1] S.M. Alsulami, P. Nevai, J. Szabados, and W. Van Assche, *A family of nonlinear difference equations: existence, uniqueness, and asymptotic behavior of positive solutions*, J. Approx. Theory 193 (2015), pp. 39–55.

[2] P.M. Bleher and A. Deaño, *Topological expansion in the cubic random matrix model*, Int. Math. Res. Not. IMRN (2013), no. 12, pp. 2699–2755.

[3] S. Bonan and P. Nevai, *Orthogonal polynomials and their derivatives. I*, J. Approx. Theory 40 (1984), pp. 134–147.

[4] P.A. Clarkson, *Painlevé equations — non-linear special functions*, in "Orthogonal Polynomials and Special Functions: Computation and Application", F. Marcellán and W. Van Assche (Editors), Lect. Notes Math., vol. 1883, pp. 331–411, Springer-Verlag, Berlin, 2006.

[5] P.A. Clarkson, *On Airy solutions of the second Painlevé equation*, arXiv:1510.08326 [nlin.SI] (2015).

[6] G. Filipuk, W. Van Assche, and L. Zhang, *Multiple orthogonal polynomials with an exponential cubic weight*, J. Approx. Theory 193 (2015), pp. 1–25.

[7] A.S. Fokas, B. Grammaticos, and A. Ramani, *From continuous to discrete Painlevé equations*, J. Math. Anal. Appl. 180 (1993), pp. 342–360.

[8] A.S. Fokas, A.R. Its, and A.V. Kitaev, *Discrete Painlevé equations and their appearance in quantum-gravity*, Comm. Math. Phys. 142 (1991), pp. 313–344.

[9] B. Fornberg and J.A.C. Weideman, *A computational exploration of the second Painlevé equation*, Found. Comput. Math. 14 (2014), pp. 985–1016.

[10] P.J. Forrester and N.S. Witte, *Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and the GUE*, Comm. Math. Phys. 219 (2001), pp. 357–398.

[11] B. Grammaticos and A. Ramani, *Discrete Painlevé equations: a review*, in "Discrete Integrable Systems", B. Grammaticos, T. Tamizhmani and Y. Kosmann-Schwarzbach (Editors), Lect. Notes Phys., vol. 644, pp. 245–321, Springer-Verlag, Berlin, 2004.

[12] E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[13] J.S. Lew and D.A. Quarles, *Nonnegative solutions of a nonlinear recurrence*, J. Approx. Theory 38 (1983), pp. 357–379.

[14] A. Magnus, *Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Comput. Appl. Math. 57 (1995), pp. 215–237.

[15] A. Magnus, *Freud's equations for orthogonal polynomials as discrete Painlevé equations*, in "Symmetries and Integrability of Difference Equations", P.A. Clarkson and F.W. Nijhoff (Editors), *London Math. Soc. Lecture Note Ser.*, vol. 255, pp. 228–243, Cambridge University Press, Cambridge, 1999.

[16] P. Nevai, *Orthogonal polynomials associated with* $\exp(-x^4)$, in "Second Edmonton Conference on Approximation Theory", CMS Conf. Proc. **3**, pp. 263–285, Amer. Math. Soc., Providence, RI, 1983.

[17] K. Okamoto, *Studies on the Painlevé equations III. Second and fourth Painlevé equations,* $\mathrm{P_{II}}$ *and* $\mathrm{P_{IV}}$, Math. Ann. 275 (1986), pp. 221–255.

[18] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark (Editors), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.